

Higher Order City Voronoi Diagrams

Andreas Gemsa¹, D. T. Lee^{2,3}, Chih-Hung Liu^{1,2}, and Dorothea Wagner¹

¹ Karlsruhe Institute of Technology, Germany

² Academia Sinica, Taiwan

³ National Chung Hsing University, Taiwan

Abstract. We investigate higher-order Voronoi diagrams in the *city metric*. This metric is induced by quickest paths in the L_1 metric in the presence of an accelerating transportation network of axis-parallel line segments. For the structural complexity of k^{th} -order city Voronoi diagrams of n point sites, we show an upper bound of $O(k(n-k) + kc)$ and a lower bound of $\Omega(n + kc)$, where c is the complexity of the transportation network. This is quite different from the bound $O(k(n-k))$ in the Euclidean metric [11]. For the special case where $k = n - 1$ the complexity in the Euclidean metric is $O(n)$, while that in the city metric is $\Theta(nc)$. Furthermore, we develop an $O(k^2(n+c)\log n)$ -time iterative algorithm to compute the k^{th} -order city Voronoi diagram and an $O(nc\log^2(n+c)\log n)$ -time divide-and-conquer algorithm to compute the farthest-site city Voronoi diagram.

1 Introduction

In many modern cities, e.g., Manhattan, the layout of the road network resembles a grid. Most roads are either horizontal or vertical, and thus pedestrians can move only either horizontally or vertically. Large, modern cities also have a public transportation network (e.g., bus and rail systems) to ensure easy and fast travel between two places. Traveling in such cities can be modeled well by the *city metric*. This metric is induced by quickest paths in the L_1 metric in the presence of an accelerating transportation network. We assume that the traveling speed on the transportation network is a given parameter $\nu > 1$. The speed while traveling off the network is 1. Further, we assume that the transportation network can be accessed at any point. Then the distance between two points is the minimum time required to travel between them.

For a given set S of n point sites (i.e., a set of n coordinates) and a transportation network in the plane, the k^{th} -order city Voronoi diagram $V_k(S)$ partitions the plane into *Voronoi regions* such that all points in a Voronoi region share the same k nearest sites with respect to the city metric.

The k^{th} -order city Voronoi diagram can be used to resolve the following situation: a pedestrian wants to know the k nearest facilities (e.g., k stores, or k hospitals) such that he can make a well-informed decision as to which facility to go to. For this kind of scenario, the k^{th} -order city Voronoi diagram provides a way to determine the k nearest facilities, by modeling the facilities as point sites.

The nearest-site (first-order) city Voronoi diagram has already been well-studied [1,4,6,10]. Its structural complexity (the size) has been proved to be $O(n + c)$ [1], where c is the

Table 1. Comparison between the Euclidean and the city metric. Our results are marked by \dagger .

	Euclidean		City	
	Structural Complexity	Time Complexity	Structural Complexity	Time Complexity
nearest-site	$\Theta(n)$	$O(n \log n)$	$\Theta(n + c)$	$O((n + c) \log(n + c))$ [6]
farthest-site	$\Theta(n)$	$O(n \log n)$	$\Theta(nc)^\dagger$	$O(nc \log n \log^2(n + c))^\dagger$
k^{th} -order	$\Theta(k(n - k))$ [11]	$O(k^2 n \log n)$ $O(n^2 + k(n - k) \log^2 n)$	upper bound: $O(k(n - k) + kc)^\dagger$ lower bound: $\Omega(n + kc)^\dagger$	$O(k^2(n + c) \log(n + c))^\dagger$

complexity of the transportation network. Such a Voronoi diagram can be constructed in $O((n + c) \log(n + c))$ time [6]. However, to the best of our knowledge there is no existing work regarding k^{th} -order or farthest-site (i.e., $(n - 1)^{\text{st}}$ -order) city Voronoi diagrams.

Contrary to k^{th} -order city Voronoi diagrams, k^{th} -order Euclidean Voronoi diagrams have been studied extensively for over thirty years. Their structural complexity has been shown to be $O(k(n - k))$ [11]. They can be computed by an iterative construction method in $O(k^2 n \log n)$ time [11] or by a different approach based on geometric duality and arrangements in $O(n^2 + k(n - k) \log^2 n)$ time [7]. Additionally, there are several randomized algorithms [2,13] and on-line algorithms [3,5].

One of the most significant differences between the Euclidean metric and the city metric that influences the computation and complexity of Voronoi diagrams is the complexity of a bisector between two points. In the Euclidean or the L_1 metric such a bisector has constant complexity, while in the city metric the complexity may be $\Omega(c)$ [1] and can even be a closed curve. Since the properties of a bisector between two points significantly affect the properties of Voronoi diagrams, a k^{th} -order city Voronoi diagram can be very different from a Euclidean one. First, this property makes it non-trivial to apply existing approaches for constructing Euclidean Voronoi diagrams to the city Voronoi diagrams. Secondly, this property also indicates that the complexity of k^{th} -order Voronoi diagrams may depend significantly on the complexity of the transportation network.

In this paper, we derive bounds for the structural complexity of the k^{th} -order Voronoi diagram and develop algorithms for computing the k^{th} -order city Voronoi diagram. The remainder of this paper is organized as follows. In Section 2, we introduce two important concepts, wavefront propagation [1] and shortest path maps [6], which are essential for the proofs in the subsequent sections. In Section 3, we adopt the wavefront propagation to introduce a novel interpretation of the iterative construction method of Lee [11], and use this interpretation to derive an upper bound of $O(k(n - k) + kc)$ for the structural complexity of k^{th} -order city Voronoi diagrams, where c is the complexity of the transportation network. Then, we construct a worst-case example to obtain a lower bound of $\Omega(n + kc)$. Finally, we extend the insights of Section 3 to develop an iterative algorithm to compute k^{th} -order city Voronoi diagrams in $O(k^2(n + c) \log(n + c))$ time (see Section 4). Moreover, we give a divide-and-conquer approach to compute farthest-site city Voronoi diagrams in $O(nc \log n \log^2(n + c))$ time. We conclude the paper in Section 5.

For an overview of our contribution and a comparison between Euclidean and city metric see Table 1.

2 Preliminaries

In this section we introduce the notation used throughout this paper for k^{th} -order city Voronoi diagrams. Then, we introduce two well-established concepts in the context of Voronoi diagrams, which are important for the proofs in the subsequent sections.

A transportation network is a planar straight-line graph $C = (V_C, E_C)$ with *isothetic* edges only, i.e., edges that are either horizontal or vertical, and all transportation edge have identical speed $\nu > 1$. We define $c := |V_C|$, and since the degree of a vertex in V_C is at most four, $|E_C|$ is $\Theta(c)$. We denote the distance of two points in the L_1 metric by d_1 and in the city metric by d_C . Similarly, we denote the bisector between two points by B_1 and B_C for the L_1 and city metric, respectively. Additionally, for the city metric we define the distance between a point $p \in \mathbb{R}^2$ and a set of points $H \subset \mathbb{R}^2$ to be $d_C(p, H) = \max_{q \in H} d_C(p, q)$. This allows us to define the bisector $B_C(H_1, H_2) = \{r \in \mathbb{R}^2 \mid d_C(r, H_1) = d_C(r, H_2)\}$ between two sets of points H_1 and H_2 .

By $V_k(H, S)$ we denote a *Voronoi region* of $V_k(S)$ associated with a k -element subset $H \subset S$. The common boundary between two adjacent Voronoi regions $V_k(H_1, S)$ and $V_k(H_2, S)$ is called a *Voronoi edge*. This Voronoi edge is a part of $B_C(H_1, H_2) = B_C(p, q)$ where $H_1 \setminus H_2 = \{p\}$ and $H_2 \setminus H_1 = \{q\}$ [11]. The common intersection among more than two Voronoi regions is called a *Voronoi vertex*. Without loss of generality, we assume that no point in the plane is equidistant from four sites in S with respect to the city metric, ensuring that the degree of a Voronoi vertex is exactly three.

Wavefront Propagation. The wavefront propagation is a well-established model to define Voronoi diagrams [1]. In Section 3, we will use this concept to interpret the formation of $V_k(S)$ and analyze its structural complexity.

For a fixed site $p \in S$, let $W_p(x) = \{q \mid q \in \mathbb{R}^2, d_C(p, q) = x\}$. This means that for a fixed $x \in \mathbb{R}_0^+$ the wavefront $W_p(x)$ is the circle centered at p with radius x . We call p the *source* of $W_p(x)$. Note that we can view $W_p(x)$ as the wavefront at time x of the wave that originated in p at time 0. We refer to such a wavefront as W_p if the value of x is unimportant.

Initially, the wavefront W_p is a diamond. When it touches a part of the transportation network for the first time it changes its propagation speed and, hence its shape; see Fig. 1. Certain points on the transportation network play an important role to determine the structural complexity of k^{th} -order city Voronoi diagrams. Thus, we introduce the following definitions. For a point $v \in \mathbb{R}^2$, let $P(v)$ denote the *isothetic projection* of v onto the transportation network, i.e., we shoot an isothetic half-ray starting at v in each of the four directions and for each half-ray we add its first intersection with an edge of the transportation network to $P(v)$. It is easy to see that there are at most four such intersections.

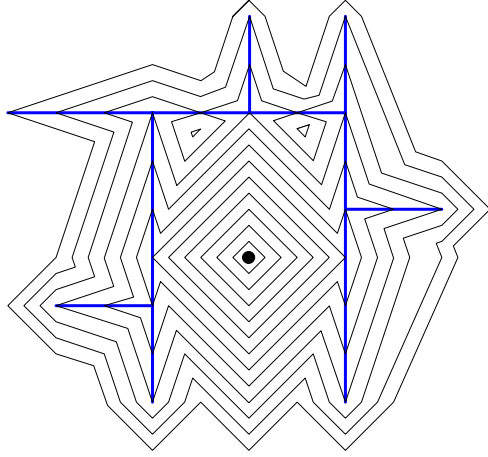


Fig. 1. Wavefront Propagation.

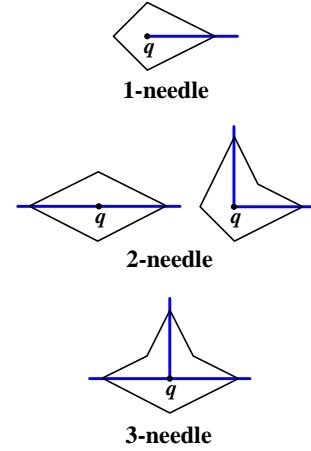


Fig. 2. 1-needle, 2-needle and 3-needle.

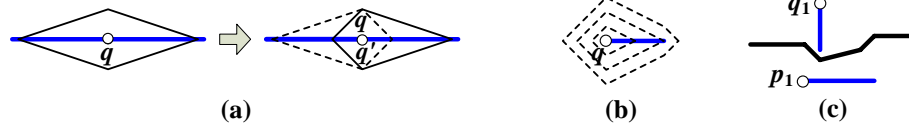


Fig. 3. (a) A 2-needle is two 1-needles. (b) Wavefront propagation of a 1-needle. (c) $B_1(\eta_p(p_1), \eta_q(q_1))$.

For a set $X \subset \mathbb{R}^2$, we denote the *isothetic projection* of the set X as $\mathcal{P}(X) = \bigcup_{v \in X} P(v)$. For a site $p \in S$, we call the set $A(p) = P(p) \cup V_C \cup \mathcal{P}(V_C) \cup \{p\}$ *activation points* (we added $\{p\}$ to the list for ease of argumentation in some of our proofs).

As shown by Aichholzer et al. [1], the wavefront W_p changes its propagation speed only if it hits a vertex in $A(p)$. Since the shape of W_p can become very complex after it hits multiple activation points, we make the following simplification for the remainder of this paper: if a wavefront W_p touches a point $q \in A(p)$ we do not change the propagation speed of W_p . Instead, we start a new wavefront at q , which, in turn, starts new wavefronts at points in $A(p)$ if it reaches them earlier than any other wavefront. Hereafter, the start of the propagation of a new wavefront is called an *activation event*, or we say a wavefront is *activated*. The shape of such a new wavefront depends on the position of q on the transportation network. It can be categorized into one of three different shapes: 1-needle, 2-needle, and 3-needle [1] (see Fig. 2). To simplify things, we treat a 2-needle (3-needle) as two (three) 1-needles (see Fig. 3(a)).

When a 1-needle reaches the end of the corresponding network segment, as shown in Fig. 3(b), its shape changes (permanently) [1]. In order to interpret the propagation of a 1-needle, Bae et al. [6] introduced a structure called *needle*. A needle $\eta_p(q, q')$ is a network segment $\overline{qq'}$ with weight $d_C(p, q)$, where $p \in S$ and $q, q' \in A(p)$. Propagating a wavefront from $\eta_p(q, q')$ is equivalent to propagating a 1-needle from q on the network segment $\overline{qq'}$.

at time $d_C(p, q)$. If q' is obvious or unimportant we may refer to $\eta_p(q, q')$ as $\eta_p(q)$. Bae et al. also defined the L_1 distance $d_1(\eta_p(q, q'), r)$ between a needle $\eta_p(q, q')$ and a point r as $d_C(p, q)$ plus the length of a quickest path from q to r accelerated by qq' . Thus, the bisector $B_1(\eta_p(p_1), \eta_q(q_1))$ between two needles $\eta_p(p_1)$ and $\eta_q(q_1)$ is well defined (see Fig. 3(c)).

Shortest Path Map. We use the wavefront model to define shortest path maps [6], and use this concept to explain the formation of *mixed vertices* in Section 3.1, which are important for deriving the structural complexity of the k^{th} -order city Voronoi diagram.

For a site $p \in S$ its shortest path map \mathcal{SPM}_p is a planar subdivision that can be obtained as follows: start by propagating a wavefront from the site p . When a point $q \in A(p)$ is touched for the first time by a wavefront, propagate an additional wavefront from $\eta_p(q)$. Eventually, each point $r \in \mathbb{R}^2$ is touched for the first time by a wavefront propagated from a needle $\eta_p(q)$, where $q \in A(p)$ and $d_1(r, \eta_p(q)) = \min_{q' \in A(p)} d_1(r, \eta_p(q'))$, and q is called the *predecessor* of r . This induces \mathcal{SPM}_p . In detail, \mathcal{SPM}_p partitions the plane into at most $|A(p)| = O(c)$ regions $\mathcal{SPM}_p(q)$ such that all points $r \in \mathcal{SPM}_p(q)$ share the same predecessor q and q is on a quickest path from p to r , i.e., $d_C(p, r) = d_C(p, q) + d_C(q, r) = d_1(r, \eta_p(q))$. As proved in [6], the common edge between $\mathcal{SPM}_p(q)$ and $\mathcal{SPM}_p(q')$ where $q, q' \in A(p)$ belongs to the bisector $B_1(\eta_p(q), \eta_p(q'))$. Fig. 4 illustrates an example of the function of shortest path maps where the two Voronoi regions of $V_1(\{p, q\})$ are partitioned by \mathcal{SPM}_p and \mathcal{SPM}_q , respectively.

3 Complexity

In this section we derive an upper and a lower bound of the structural complexity of the k^{th} -order city Voronoi diagram $V_k(S)$. In Section 3.1, we first introduce a special degree-2 vertex on a Voronoi edge called *mixed vertex* which is similar to the mixed Voronoi vertices of Cheong et al. [8] for farthest-polygon Voronoi diagrams. Then we derive an upper bound of the structural complexity of $V_k(S)$ in terms of the number of mixed vertices and Voronoi regions. In Section 3.2, we adopt the wavefront concept to introduce a new interpretation for the iterative construction of $V_k(S)$ by Lee [11]. This yields an upper bound for the structural complexity of $V_k(S)$. In Section 3.3 we construct a worst-case example to obtain a lower bound for the structural complexity of $V_k(S)$.

3.1 Mixed Vertices

Definition 1 (Mixed Vertex) For two sites $p, q \in S$ and a Voronoi edge e which is part of $B_C(p, q)$, a point r on e is a mixed vertex if there are $p_1, p_2 \in A(p)$ and $q_1 \in A(q)$ such that $r \in \mathcal{SPM}_p(p_1) \cap \mathcal{SPM}_p(p_2) \cap \mathcal{SPM}_q(q_1)$.

For instance, Fig. 4 shows a first-order city Voronoi diagram $V_1(\{p, q\})$, where the mixed vertices are marked with a square and denoted by m_1, \dots, m_4 . The vertex m_2 is a

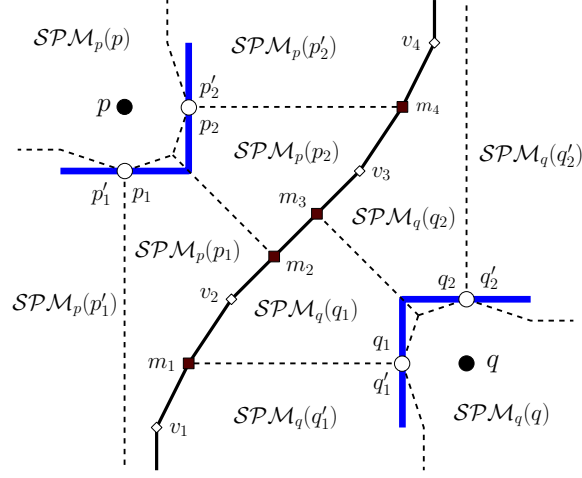


Fig. 4. $B_C(p, q)$ (solid thin edge), where m_1, \dots, m_4 are mixed vertices.

mixed vertex because it is in $\text{SPM}_p(p_1) \cap \text{SPM}_p(p_2) \cap \text{SPM}_q(q_1)$. Definition 1 yields the following.

Lemma 1 *If a Voronoi edge e contains $m \geq 0$ mixed vertices, its complexity is $O(m + 1)$.*

Proof. Suppose e is part of a bisector $B_C(p, q)$, where $p, q \in S$. Consider two consecutive mixed vertices m_1 and m_2 on $B_C(p, q)$, where $m_1 \in \text{SPM}_p(p_1) \cap \text{SPM}_q(q'_1) \cap \text{SPM}_q(q_1)$ and $m_2 \in \text{SPM}_p(p_1) \cap \text{SPM}_p(p_2) \cap \text{SPM}_q(q_1)$ (see Fig. 4). Consider each point v on $B_C(p, q)$ between m_1 and m_2 . Since v belongs to $\text{SPM}_p(p_1) \cap \text{SPM}_q(q_1)$, we have $d_C(v, p) = d_C(v, p_1) + d_C(p_1, p) = d_1(v, \eta_p(p_1))$ and $d_C(v, q) = d_C(v, q_1) + d_C(q_1, q) = d_1(v, \eta_q(q_1))$. Together with $d_C(v, p) = d_C(v, q)$, v belongs to $B_1(\eta_p(p_1), \eta_q(q_1))$ (recall Fig 3(c)). As a result, if a Voronoi edge e contains m mixed Voronoi vertices, e consists of $m + 1$ parts, each of which belongs to a bisector between two needles. Since the complexity of an L_1 bisector between two needles is $O(1)$ [4], the complexity of e is $O(m + 1)$. Note that the complexity of a bisector between two points in the city metric is $\Theta(c)$, while the complexity of an L_1 bisector between two needles is $O(1)$. \square

Lemma 2 *An upper bound for the structural complexity of a k^{th} -order city Voronoi diagram $V_k(S)$ is $O(M + k(n - k))$, where M is the total number of mixed vertices.*

Proof. Lee [11] proved that the number of Voronoi regions in the k^{th} -order Voronoi diagram is $O(k(n - k))$ in any distance metric satisfying the triangle inequality, and so is the number of Voronoi edges. By Lemma 1, if a Voronoi edge e contains m_e mixed Voronoi vertices, its complexity is $O(m_e + 1)$. Suppose a city Voronoi diagram $V_k(S)$ contains a set E of Voronoi edges, and each edge $e \in E$ contains m_e mixed Voronoi vertices. Then, the complexity of all edges, i.e., the structural complexity of $V_k(S)$, is $\sum_{e \in E} O(m_e + 1) = O(M + |E|)$. Since $|E| = O(k(n - k))$, it follows that $O(M + |E|) = O(M + k(n - k))$. \square

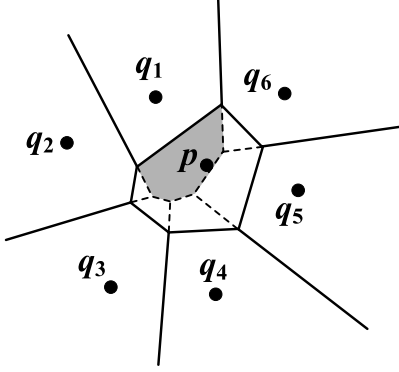


Fig. 5. $V_1(H, S) \cap V_1(Q) = V_1(H, S) \cap V_2(S)$ where $H = \{p\}$, $Q = \bigcup_{1 \leq i \leq 6} \{q_i\}$, and $S = H \cup Q$.

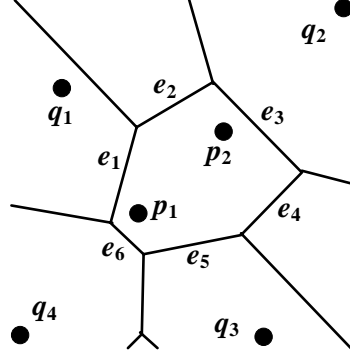


Fig. 6. $V_2(H, S)$ where $H = \{p_1, p_2\}$, $Q = \{q_1, q_2, q_3, q_4\}$, and $S = H \cup Q$.

For the proof in Section 3.2, we further categorize the mixed vertices. Let m be a mixed vertex on the Voronoi edge between $V_k(H_1, S)$ and $V_k(H_2, S)$, where $H_1 \setminus H_2 = \{p\}$ and $H_2 \setminus H_1 = \{q\}$. We call m an *interior mixed vertex* of $V_k(H_1, S)$ if $m \in \mathcal{SPM}_p(p_1) \cap \mathcal{SPM}_p(p_2) \cap \mathcal{SPM}_q(q_1)$, for some $p_1, p_2 \in A(p)$ and $q_1 \in A(q)$; otherwise, we call m an *exterior mixed vertex* of $V_k(H_1, S)$. For example, in Fig.4 the vertices m_2 and m_4 both are interior mixed vertices of $V_1(\{p\}, \{p, q\})$ and exterior mixed vertices of $V_1(\{q\}, \{p, q\})$.

3.2 Upper Bound

Throughout this subsection, we consider a Voronoi region $V_j(H, S)$ of a j^{th} -order Voronoi diagram $V_j(S)$, where $H \subset S$ and $|H| = j$. Let $V_j(H, S)$ have h_H adjacent Voronoi regions $V_j(H_i, S)$ for $1 \leq i \leq h_H$. Note that the subsets H_i and H differ in exactly one element [11]. In the following let $H_i \setminus H = \{q_i\}$, $Q = \{q_1, \dots, q_{h_H}\}$, and $\ell_H = |Q|$.

Lee [11] proved that in any distance metric satisfying the triangle inequality, $V_j(H, S) \cap V_1(Q) = V_j(H, S) \cap V_{j+1}(S)$, and thus computing $V_1(Q)$ for all the Voronoi regions $V_j(H, S)$ of $V_j(S)$ yields $V_{j+1}(S)$, leading to an iterative construction for $V_k(S)$ for any $k < n$. Fig. 5 illustrates this iteration technique for the Euclidean metric: solid segments form $V_1(H, S)$ and dashed segments form $V_1(Q)$. Since the gray region is part of $V_1(\{p\}, S)$ and also part of $V_1(\{q_1\}, Q)$, all points in the gray region share the same two nearest sites p and q_1 , implying that the gray region is part of $V_2(\{p, q_1\}, S)$.

We adopt wavefront propagation to interpret this iterative construction in a new way, which will lead to the main proof of this section. Let us imagine that a wavefront is propagated from each site $q \in Q$ into the Voronoi region $V_j(H, S)$. If a point $r \in \mathbb{R}^2$ is first touched by the wavefront that propagated from q , r belongs to $V_{j+1}(H \cup \{q\}, S)$.

Note that when $j \geq 2$, $|Q|$ is not necessarily the number of adjacent regions, i.e., $\ell_H \leq h_H$. Fig. 6 illustrates an example for the Euclidean metric: $V_2(H, S)$ has 6 adjacent Voronoi

regions but $|Q| = \ell_H$ is only 4. This is because for a site $q \in Q$, $B_2(\{q\}, H) \cap V_j(H, S)$ may consist of more than one Voronoi edge, where $B_2(\{q\}, H)$ is a Euclidean bisector between $\{q\}$ and H (similar to $B_C(H_1, H_2)$ defined in Section 2). For instance, as shown in Fig. 6, $e_{q_1} = B_2(\{q_1\}, H) \cap V_2(H, S)$ consists of two Voronoi edges e_1 and e_2 .

Now we transfer our new interpretation to the city metric. Let e_q be $B_C(\{q\}, H) \cap V_j(H, S)$ for some site $q \in Q$. If e_q contains m_q **exterior** mixed vertices with respect to $V_j(H, S)$, e_q intersects $m_q + 1$ regions in \mathcal{SPM}_q . We denote these regions by $\mathcal{SPM}_q(v_z)$ for $1 \leq z \leq m_q + 1$. Note that all v_z must be in $A(q)$. Then, instead of propagating a single wavefront from q into $V_j(H, S)$ (as in the Euclidean metric), we propagate $m_q + 1$ wavefronts, namely one from each $\eta_q(v_z)$ into $V_j(H, S)$.

As a result, if $V_j(H, S)$ contains m_H exterior mixed vertices, $m_H + \ell_H$ wavefronts will be propagated into $V_j(H, S)$. During the process, when a point $r \in V_j(H, S)$ is *first* touched by a wavefront propagated from $\eta_q(v)$, $q \in Q$ and $v \in A(q)$, we propagate a new wavefront from $\eta_q(r)$, i.e., an activation event occurs, if (i) $r \in \mathcal{P}(V_C) \cup V_C$ or (ii) $v = q$ and $r \in \mathcal{P}(q)$. These two conditions amount to $r \in A(q) \setminus \{q\}$, but this classification will help us to derive the number of mixed vertices. This is due to the fact that during the $k - 1$ iterations for computing $V_{j+1}(S)$ from $V_j(S)$ for $1 \leq j \leq k - 1$, $\mathcal{P}(V_C)$ contributes $O(kc)$ activation events, but $\mathcal{P}(S)$ only contributes $O(n)$.

Lemma 3 *If $V_j(H, S)$ contains m_H exterior mixed vertices, then $V_j(H, S) \cap V_{j+1}(S)$ contains at most $m_H + 2c_H + 2a_H$ mixed vertices, where $c_H = |(\mathcal{P}(V_C) \cup V_C) \cap V_j(H, S)|$ and a_H is the number of activation events associated with points in $\mathcal{P}(S)$.*

Proof. According to the above discussion, we propagate $m_H + \ell_H$ wavefronts into $V_j(H, S)$. All those wavefronts combined generate at most c_H new wavefronts from points in $\mathcal{P}(V_C) \cap V_j(H, S)$, and a_H new wavefronts from points in $\mathcal{P}(S) \cap V_j(H, S)$. Note that $c_H = |(\mathcal{P}(V_C) \cup V_C) \cap V_j(H, S)|$ (condition (i)) but $a_H \leq |\mathcal{P}(S) \cap V_j(H, S)|$ (condition (ii)). Let W be the set of the $m_H + \ell_H + c_H + a_H$ wavefronts. For each point $r \in V_j(H, S)$, if r is first touched by a wavefront $w \in W$ it is associated with w . This will partition $V_j(H, S)$ into $m_H + \ell_H + c_H + a_H$ regions. We view those regions as a special Voronoi diagram $V_1(W)$. Note that $m_H + \ell_H$ of those regions are unbounded.

$V_j(H, S) \cap V_{j+1}(S)$ is a subgraph of $V_1(W)$ since if a point $r \in V_j(H, S)$ is first touched by a wavefront in W propagated from $\eta_q(v)$, r belongs to $V_{j+1}(H \cup \{q\}, S)$. Without loss of generality, we assume every vertex of $V_1(W)$ has degree 3. According to Euler's formula it holds that $N_V = 2(N_R - 1) - N_U$, where N_V , N_R and N_U are the numbers of vertices, regions, and unbounded regions, respectively. Since $V_1(W)$ contains $m_H + \ell_H$ unbounded regions and $m_H + \ell_H + c_H + a_H$ bounded regions, $V_1(W)$ contains $m_H + \ell_H + 2c_H + 2a_H - 2$ vertices. By [11], since $|Q| = \ell_H$, there are $\ell_H - 2$ Voronoi vertices in $V_{j+1}(S) \cap V_j(H, S)$. Therefore, $V_j(H, S) \cap V_{j+1}(S)$ contains at most $(m_H + \ell_H + 2c_H + 2a_H - 2) - (\ell_H - 2) = m_H + 2c_H + 2a_H$ mixed vertices. \square

Applying Lemma 3 to each region of $V_j(S)$, yields a recursive formula for the total number of mixed vertices m_{j+1} in $V_{j+1}(S)$: $m_{j+1} = m_j + O(c) + a_j$ (see Lemma 4). In Lemma 5 we show that this formula can be bounded by $O(n + kc)$ for k iterations of this iterative approach. Finally, in Theorem 1 we combine the insights of Lemma 2 and Lemma 5 to give an upper bound for the structural complexity of $V_k(S)$.

Lemma 4 $V_{j+1}(S)$ contains $m_j + O(c) + 2a_j$ mixed vertices where m_j is the number of mixed vertices of $V_j(S)$ and a_j is the number of activation events associated with points in $\mathcal{P}(S)$ during the computation of $V_{j+1}(S)$ from $V_j(S)$.

Proof. For a Voronoi region $V_j(H, S)$, let m_H be the number of its exterior mixed vertices, let c_H be $|V_j(H, S) \cap (\mathcal{P}(V_C) \cup V_C)|$ and let a_H be number of activation events associated with vertices in $\mathcal{P}(S) \cap V_j(H, S)$ during the computation. If $V_j(H, S)$ is empty, $m_H = c_H = a_H = 0$. By Lemma 3, $V_j(H, S) \cap V_{j+1}(S)$ contains at most $m_H + 2c_H + 2a_H$ mixed Voronoi vertices. Therefore, the total number of mixed vertices of $V_{j+1}(S)$ is bounded by:

$$\sum_{H \in S, |H|=j} (m_H + 2c_H + 2a_H) = m_j + 2|\mathcal{P}(V_C)| + 2a_j.$$

□

Lemma 5 The number of mixed vertices of $V_k(S)$ is $O(n + kc)$.

Proof. Let m_j be the total number of mixed Voronoi vertices of $V_j(S)$ and let a_j be the number of activation events associated with vertices in $\mathcal{P}(S)$ during the computation of $V_{j+1}(S)$ from $V_j(S)$, described by our algorithm. Then, by Lemma 4 the following holds:

$$m_k = m_{k-1} + O(c) + a_{k-1} = \dots = m_1 + O(kc) + 2 \sum_{1 \leq j \leq k-1} a_j.$$

Now, we show an upper bound for the complexity of $\sum_{1 \leq j \leq k-1} a_j$. For a vertex $v \in P(q)$ where $q \in S$, let the j^{th} iteration be the first time when v is activated by a wavefront propagated from q , i.e. $\eta_q(v)$ will propagate a wavefront, and let v belong to $V_j(H, S)$. Due to this and since the points in H are the j nearest sites of v , q is the $(j+1)^{\text{st}}$ nearest site of v . Therefore, for $j' > j$, if $v \in V_{j'}(H', S)$, $q \in H'$, implying that v will not be activated by a wavefront propagated from q again after the j^{th} iteration. In other words, v causes at most one activation event due to the wavefront propagation of q during the $k-1$ iterations, and the $\mathcal{P}(S)$ causes $O(n)$ activation events, i.e., $\sum_{1 \leq j \leq k-1} a_j = O(n)$. Furthermore, m_1 has been proved to be $O(n + c)$ [1,6,10]. Therefore, $m_k = O(n + kc)$. □

Theorem 1 The structural complexity of $V_k(S)$ is $O(k(n - k) + kc)$.

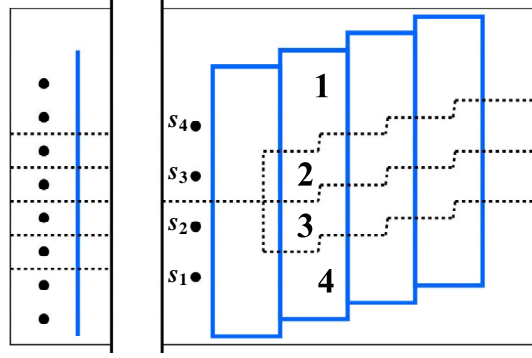


Fig. 7. This worst-case example (here with $k = 3$, $n = 12$, $c = 18$) leads to a lower bound of $\Omega(n + kc)$. The bold solid segments depict the transportation network, and the dashed segments compose $V_k(S)$. The right part is also the farthest-site city Voronoi diagram of $\{s_1, s_2, s_3, s_4\}$, where all points in Region i share the same farthest site s_i .

3.3 Lower Bound

We construct a worst-case example (see Fig. 7) to derive a lower bound for the structural complexity of the k^{th} -order city Voronoi diagram $V_k(S)$. The example consists of a left part and a right part which are placed with a sufficiently large distance between them. We place one vertical network segment in the left part and build a stairlike transportation network in the right part. Then, we place $k + 1$ sites in the right part and the remaining $n - k - 1$ sites in the left part. Since the distance between the left and right part is extremely large, the $n - k - 1$ sites in the left part hardly influence the formation of $V_k(S)$ in the right part. Therefore, $V_k(S)$ in the right part forms the farthest-site city Voronoi diagram of the $k + 1$ sites, because sharing the same k nearest sites among $k + 1$ sites is equivalent to sharing the same farthest site among the $k + 1$ sites.

By construction, as shown in the right part of Fig. 7, all the points in Region i share the same farthest site s_i . Since we can set the speed ν to be large enough, for each point x in Region 2, the shortest path between x and s_1 (s_2) moves along the transportation network counterclockwise, and thus $d_C(s_2, x) > d_C(s_1, x)$. The common Voronoi edge between Regions i and $(i + 1)$ contains at least $(\frac{c-2}{4} - 1) \cdot 2 + 1$ (here: 7) segments since the transportation network forms $\frac{c-2}{4}$ rectangles and each rectangle except the first one contains two vertices of the Voronoi edge. Therefore, in the right part, $V_k(S)$ contains at least $(k - 1) \frac{c-6}{2} = \Omega(kc)$ segments. Together with the $\Omega(n - k)$ in the left part, we obtain the following lower bound.

Theorem 2 *The structural complexity of $V_k(S)$ is $\Omega(n + kc)$.*

Proof. We need to distinguish two cases:

i) $n - k - 1 \geq k + 1 \Rightarrow 2k \leq n - 2$: This implies that the left part contains $n - k - 2$ segments, and thus, $V_k(S)$ contains $n - k - 2 + \Omega(kc) = \Omega(n + kc)$ segments.

ii) $2k > n - 2$: This implies that the left part is empty, and thus, $V_k(S)$ contains $\Omega(kc) = \Omega(k(c + 2)) = \Omega(n + kc)$ segments.

This concludes the proof. \square

4 Algorithms

In this section we present an iterative algorithm to compute k^{th} -order city Voronoi diagrams in $O(k^2(n + c) \log(n + c))$ time. Its main idea has already been introduced in the complexity considerations in Section 3.2. For the special case of the farthest-site Voronoi diagram, i.e., the $(n - 1)^{\text{st}}$ -order Voronoi diagram, this algorithm takes $O(n^2(n + c) \log(n + c))$ time. However, for the farthest-site city Voronoi diagram we present a divide-and-conquer algorithm which requires only $O(nc \log^2(n + c) \log n)$ time.

4.1 Iterative Algorithm for k^{th} -Order City Voronoi Diagrams

We describe an algorithm to compute k^{th} -order city Voronoi diagrams $V_k(S)$ based on the ideas in Section 3.2 and Bae et al.'s [6] $O((n + c) \log(n + c))$ -time algorithm for the first-order city Voronoi diagram $V_1(S)$. Bae et al.'s approach views each point site in S as a needle with zero-weight and zero-length, and simulates the wavefront propagation from those needles to compute $V_1(S)$. Since their approach can handle general needles, we adopt it to simulate the wavefront propagation of Section 3.2 to compute $V_{j+1}(S)$ from $V_j(S)$.

Algorithm. We give the description of our algorithm for a single Voronoi region $V_j(H, S)$. All four steps have to be repeated for each Voronoi region of $V_j(S)$.

Let $V_j(H, S)$ have h adjacent regions $V_j(H_i, S)$ with $H_i \setminus H = \{q_i\}$ for $1 \leq i \leq h$ and let $Q = \bigcup_{1 \leq i \leq h} q_i$. Our algorithm computes $V_j(H, S) \cap V_{j+1}(S)$ as follows:

1. Compute a new set N of sites (needles): For $1 \leq i \leq h$, if the Voronoi edge between $V_j(H_i, S)$ and $V_j(H, S)$ intersect m_i regions $\mathcal{SPM}_{q_i}(v_z)$ in \mathcal{SPM}_{q_i} , $1 \leq z \leq m_i$, insert every $\eta_{q_i}(v_z)$ into N .
2. Construct a new transportation network C_H from C : For each point $v \in (\mathcal{P}(V_C) \cup \mathcal{P}(Q) \cup V_C) \cap V_j(H, S)$, if v is located on an edge e of C , insert e into C_H .
3. Perform Bae et al.'s wavefront-based approach to compute $V_1(N)$ under the new transportation network C_H . The approach can intrinsically handle needles as weighted sites.
4. Determine $V_j(H, S) \cap V_{j+1}(S)$ from $V_1(N)$: Consider each edge e in $V_j(H, S) \cap V_1(N)$. Let e be an edge between $V_1(\eta_p(v_p), N)$ and $V_1(\eta_q(v_q), N)$ where $p, q \in S$, $v_p \in A(p)$ and $v_q \in A(q)$. If $p \neq q$, then $e \cap V_j(H, S)$ is part of $V_j(H, S) \cap V_{j+1}(S)$.

Note that Step 2 is used only to reduce the runtime of the algorithm. Lemma 6 shows the correctness and the run time of this algorithm for a single Voronoi region.

Lemma 6 $V_j(H, S) \cap V_{j+1}(S)$ can be computed in $O((h + m + c_H) \log(n + c))$ time, where h is the number of Voronoi edges, m is the number of mixed vertices, and $c_H = |(\mathcal{P}(V_C) \cup V_C) \cap V_j(H, S)|$.

Proof. We begin by proving correctness. Since $V_j(H, S) \cap V_{j+1}(S)$ is exactly $V_j(H, S) \cap V_1(Q)$ [11], it is sufficient to prove that the algorithm correctly computes $V_j(H, S) \cap V_1(Q)$. If the algorithm fails to compute $V_1(q, Q) \cap V_j(H, S)$, $q \in Q$, it must fail to propagate a wavefront from a needle $\eta_q(v)$, where v belongs to $A(q)$ and $\mathcal{SPM}_q(v) \cap V_1(q, Q) \cap V_j(H, S)$ is nonempty. We prove that this cannot occur by contradiction. Assume that the algorithm does not propagate a wavefront from an $\eta_q(v)$ for some v in $A(q)$ and $\mathcal{SPM}_q(v) \cap V_1(q, Q) \cap V_j(H, S)$ is nonempty. However, either $v \notin V_j(H, S)$ and $\mathcal{SPM}_q(v) \cap B_C(q, p) \cap V_j(H, S)$ must be nonempty, then, Step 1 will include $\eta_q(v)$ in N . Or $v \in V_j(H, S)$, then Step 2 will include the corresponding network segment in C_H , and thus $\eta_q(v)$ will be activated to propagate a wavefront. Both possibilities contradict the initial assumption. Therefore, the algorithm correctly computes $V_j(H, S) \cap V_{j+1}(S)$.

We proceed by giving time complexity considerations. It is clear that $|N|$ is $O(m + h)$. The run time of step 1 is linear in the complexity of the boundary of $V_j(H, S)$ and thus is $O(m + h)$. Since by definition $|\mathcal{P}(V_C) \cup V_C \cap V_j(H, S)| = O(c_H)$ and $|\mathcal{P}(Q)| = O(h)$, both $|V_{C_H}|$ and $|E_{C_H}|$ are in $O(c_H + h)$. Since $|N| = O(m + h)$ and $E_{C_H} = O(c_H + h)$, Step 3 takes $O((h + m + c_H) \log(h + m + c_H))$ time [6]. Step 4 takes the time linear in the complexity of $V_1(N) \cap V_{j+1}(S)$. The activation events associated with vertices in $\mathcal{P}(S)$ are only associated to vertices in $\mathcal{P}(Q)$, we know that $|\mathcal{P}(Q)| = O(h)$. Therefore, since there are $O(m + h)$, $O(c_H)$, and $O(h)$ wavefronts due to N , $(\mathcal{P}(V_C) \cup V_C) \cap V_j(H, S)$, and $\mathcal{P}(Q)$, respectively, the complexity of $V_1(N)$ is $O(m + h) + O(c_H) + O(h) = O(h + m + c_H)$. Since $m = O(n + kc) = O(nc)$ it holds that $O(\log(h + m + c_H)) = O(\log(nc)) = O(\log(n + c)^2) = O(\log(n + c))$. We conclude that the total running time is $O((h + m + c_H) \log(n + c))$. \square

Applying Lemma 6 to each region of $V_j(S)$ combined with Theorem 1 leads to Lemma 7. The summation of $O((j(n - j) + jc) \log(n + c))$ in Lemma 7 for $1 \leq j \leq k - 1$ gives Theorem 3.

Lemma 7 $V_{j+1}(S)$ can be computed from $V_j(S)$ in $O((j(n - j) + jc) \log(n + c))$ time.

Proof. For a Voronoi region $V_j(H, S)$, let h_H be the number of Voronoi edges, m'_H be the number of mixed vertices, and c_H be $|V_j(H, S) \cap (\mathcal{P}(V_C) \cup V_C)|$. By Lemma 6, the time complexity of computing $V_{j+1}(S)$ from $V_j(S)$ is

$$\sum_{H \in S, |H|=j} ((h_H + m'_H + c_H) \log(n + c)).$$

By Theorem 1, $\sum_{H \in S, |H|=j} h_H + m'_H = O(j(n - j) + jc)$. It is also clear that $\sum_{H \in S, |H|=j} c_H = O(c)$. Therefore, the total time complexity is $O((j(n - j) + jc) \log(n + c))$. The correctness follows from the correctness proof of Lemma 6. \square

Theorem 3 $V_k(S)$ can be computed in $O(k^2(n + c) \log(n + c))$ time.

Proof. By Lemma 7, the total time complexity is $\sum_{i=1}^{k-1} O((i(n - i) + ic) \log(n + c)) = O(k^2(n + c) \log(n + c))$. \square

4.2 Divide-and-Conquer Algorithm for Farthest-Site City Voronoi Diagram

In this section we describe a divide-and-conquer approach to compute the farthest-site city Voronoi diagram $\mathcal{FV}(S)$. Since there are n Voronoi regions in $\mathcal{FV}(S)$ and each of them is associated with a site $p \in S$, we denote such a region by $\mathcal{FV}(p, S)$.

The idea behind this algorithm is as follows: To compute $\mathcal{FV}(S)$, divide S into two equally-sized sets S_1 and $S_2 = S \setminus S_1$, compute $\mathcal{FV}(S_1)$ and $\mathcal{FV}(S_2)$, and then merge the two diagrams into $\mathcal{FV}(S)$. Now, suppose we have already computed $\mathcal{FV}(S_1)$ and $\mathcal{FV}(S_2)$. Then, the edges of a Voronoi region $\mathcal{FV}(p, S)$ in $\mathcal{FV}(S)$ stem from three sources: i) contributed by $\mathcal{FV}(S_1)$, ii) contributed by $\mathcal{FV}(S_2)$, and iii) contributed by two points, one in S_1 and the other in S_2 , that have the same distance to two farthest site. In fact, the union of all of the third kind of edges is $B_C(S_1, S_2)$. Each connected component of $B_C(S_1, S_2)$ is called a *merge curve*. A merge curve can be either a closed or open simple curve.

If all the merge curves are computed, merging $\mathcal{FV}(S_1)$ and $\mathcal{FV}(S_2)$ takes time linear in the complexity of $B_C(S_1, S_2)$. To compute the merge curves, we first need to find a point on each merge curve, and then trace out the merge curves from these discovered points.

In order to compute a merge curve, we modify Cheong et al.'s divide-and-conquer algorithm [8] for farthest-polygon Voronoi diagrams in the Euclidean metric to satisfy our requirements. Given a set \mathcal{P} of disjoint polygons, $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$, of total complexity n , the farthest-polygon Voronoi diagram $\mathcal{FV}(\mathcal{P})$ partitions the plane into Voronoi regions such that all points in a Voronoi region share the same farthest polygon in \mathcal{P} . Let $|P|$ be the number of vertices of a polygon $P \in \mathcal{P}$ and let $|\mathcal{P}|$ be $\sum_{P \in \mathcal{P}} |P| = n$.

Their algorithm computes the medial-axis $\mathcal{M}(P)$ for each polygon $P \in \mathcal{P}$ and refines $\mathcal{FV}(P, \mathcal{P})$ by $\mathcal{M}(P)$. $\mathcal{M}(P)$ partitions the plane into regions such that all points in a region share the same closest element of P , where an element is a vertex or an edge of P . In other words, for each point $v \in \mathbb{R}^2$, $\mathcal{M}(P)$ provides a shortest path between v and P . Therefore, the medial axes for $\mathcal{FV}(P, \mathcal{P})$, with $P \in \mathcal{P}$, have the same function as the shortest path maps \mathcal{SPM}_p , with $p \in S$ in the city metric. By replacing \mathcal{P} and $\mathcal{M}(P)$ with S and \mathcal{SPM}_p respectively, the divide-and-conquer algorithm of Cheong et al. [8] can be modified to compute $\mathcal{FV}(S)$ with respect to the city metric.

Cheong et al. [8] pointed out the bottleneck with respect to running time is to find for each closed merge curve a point that lies on it. In order to overcome the bottleneck, the authors use some specific point location data structures [9,12]. Let \mathcal{P} be divided into two sets \mathcal{P}_1 and $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$, where $|\mathcal{P}_1| \approx |\mathcal{P}_2| \approx \frac{n}{2}$. Cheong et al. [8] construct the point location data structures for $\mathcal{FV}(\mathcal{P}_1)$ and $\mathcal{FV}(\mathcal{P}_2)$. For each polygon $P \in \mathcal{P}_1$ and each vertex $v \in \mathcal{M}(P) \cap \mathcal{FV}(P, \mathcal{P}_1)$, they perform a point location query in $\mathcal{FV}(\mathcal{P}_1)$ and $\mathcal{FV}(\mathcal{P}_2)$ (likewise for each polygon $P' \in \mathcal{P}_2$). Each point location query requires $O(\log n)$ primitive operations, and each operation tests for $O(1)$ points and takes $O(\log n)$ time. Hence, one point location query takes $O(\log^2 n)$ time. Since $|\mathcal{FV}(\mathcal{P}_1)| = |\mathcal{FV}(\mathcal{P}_2)| = O(n)$, merging $\mathcal{FV}(\mathcal{P}_1)$ and $\mathcal{FV}(\mathcal{P}_2)$ takes $O(n \log^2 n)$ time.

Since in our case $|\mathcal{FV}(S_1)| = |\mathcal{FV}(S_2)| = O(nc)$, we perform $O(nc)$ point location queries, each of which takes $O(\log^2 nc) = O(\log^2(n+c)^2) = O(\log^2(n+c))$ time. Therefore, merging $\mathcal{FV}(S_1)$ and $\mathcal{FV}(S_2)$ takes $O(nc \log^2(n+c))$ time. We conclude:

Theorem 4 $\mathcal{FV}(S)$ can be computed in $O(nc \log n \log^2(n+c))$ time.

Proof. In the beginning, for each site $p \in S$, $\mathcal{FV}(\{p\})$ is exactly \mathcal{SPM}_p . Computing \mathcal{SPM}_p takes $O(c \log c)$ time [6], implying that computing $\mathcal{FV}(\{p\})$, for all $p \in S$, takes $O(nc \log c)$ time. Consider the merge process at some level i . The set S is divided into 2^i subsets, and each of them contains at most $n/2^i$ sites. Therefore, the merging process at level i takes $2^i \cdot O(n/2^i \log^2(n/2^i + c)) = O(nc \log^2(n+c))$ time. Since there are $\log n$ levels, $\mathcal{FV}(S)$ can be computed in $O(nc \log n \log^2(n+c))$ time. \square

5 Conclusion

We contribute two major results for the k^{th} -order city Voronoi diagram. First, we prove that its structural complexity is $O(k(n-k) + kc)$ and $\Omega(n + kc)$. This is quite different from the $O(k(n-k))$ bound in the Euclidean metric [11]. It is especially noteworthy that when $k = n - 1$, i.e., the farthest-site Voronoi diagram, its structural complexity in the Euclidean metric is $O(n)$, while in the city metric it is $\Theta(nc)$. Secondly, we develop the first algorithms that compute the k^{th} -order city Voronoi diagram and the farthest-site Voronoi diagram. Our algorithms show that traditional techniques can be applied to the city metric. Furthermore, since the complexity of the first-order city Voronoi diagram is $O(n + c)$, one may think that the complexity the transportation network contributes to the complexity of the k^{th} -order city Voronoi diagram is independent of k . However, our results show that the impact of the transportation network increases with the value of k rather than being constant.

Acknowledgment

A. Gemsa received financial support by the *Concept for the Future* of KIT within the framework of the German Excellence Initiative. D. T. Lee and C.-H. Liu are supported by the National Science Council, Taiwan under grants No. NSC-98-2221-E-001-007-MY3 and No. NSC-99-2911-I-001-506.

References

1. O. Aichholzer, F. Aurenhammer, and B. Palop, “Quickest paths, straight skeletons, and the city Voronoi diagram,” *Discrete Comput. Geom.*, vol. 31, pp. 17–35, 2004.
2. P. K. Agarwal, M. de Berg, J. Matoušek, and O. Schwarzkopf, “Constructing levels in arrangements and higher order Voronoi diagrams,” *SIAM J. Comput.*, Vol. 27, No.3, pp. 654–667, 1998.

3. F. Aurenhammer and O. Schwarzkopf, "A simple on-line randomized incremental algorithm for computing higher order Voronoi diagrams," *Internat. J. Comput. Geom. Appl.*, Vol. 2, pp. 363–381, 1992.
4. S. W. Bae and K.-Y. Chwa, "Shortest paths and Voronoi diagrams with transportation networks under general distances," *Proc. Internat. Symp. on Algorithm and Comput.*, pp. 1007–1018, 2005.
5. J. D. Boissonnat, O. Devillers, and M. Teillaud, "A semidynamic construction of higher-order Voronoi diagrams and its randomized analysis," *Algorithmica*, vol. 9, pp. 329–356, 1993.
6. S. W. Bae, J.-H. Kim, K.-Y. Chwa, "Optimal construction of the city Voronoi diagram," *Internat. J. Comput. Geom. Appl.*, vol. 19, no. 2, pp. 95–117, 2009.
7. B. Chazelle and H. Edelsbrunner, "An improved algorithm for constructing k th-order Voronoi diagrams," *IEEE Transactions on Computers*, vol. 36, No.11, pp. 1349–1454, 1987.
8. O. Cheong, H. Everett, M. Glisse, J. Gudmundsson, S. Hornus, S. Lazard, M. Lee, and H.-S. Na, "Farthest-polygon Voronoi diagrams," *Comput. Geom. Theory Appl.*, vol. 44, pp. 234–247, 2011.
9. H. Edelsbrunner, L. J. Guibas, and J. Stolfi, "Optimal point location in a monotone subdivision," *SIAM J. Comput.*, vol. 15, no. 2, pp. 317–340, 1986.
10. R. Görke, C.-S. Shin, and A. Wolff, "Constructing the city Voronoi diagram faster," *Internat. J. Comput. Geom. Appl.*, vol. 18, no. 4, pp. 275–294, 2008.
11. D. T. Lee, "On k -nearest neighbor Voronoi diagrams in the plane," *IEEE Trans. Comput.*, vol. 31, no. 6, pp. 478–487, 1982.
12. K. Mulmuley, "A fast planar partition algorithm, I," *J. Symbolic Comput.*, vol. 10, no. 3–4, pp. 253–280, 1990.
13. K. Mulmuley, "On levels in arrangements and Voronoi diagrams," *Discrete Comput. Geom.*, vol. 6, pp. 307–338, 1991.